On the Asymptotic Solutions of Linear Differential Equations.

By Clyde E. Love.

Asymptotic developments for the irregular integrals of a linear differential equation have been obtained by Horn,[*] provided that the coefficients of the equation are themselves developable in asymptotic (or convergent) power series in the vicinity of the irregular point in question, and provided also that the roots of the so-called characteristic equation are distinct.[†] The important special case in which the point is a *regular* singular point[‡] of the differential equation has been studied by Bôcher[§] for the equation of second order, and later by Dunkel[||] for the equation of arbitrary order. But no discussion of the general problem including the various cases of repeated roots has as yet been attempted.

By way of approach to the general solution it seems worth while to consider in some detail the cases of repeated roots for various equations of special orders. Such a study is undertaken in the following pages. We restrict ourselves, for simplicity, to equations of the second and of the third order, the method used being applicable at once to equations of any order. The irregular point is taken at infinity, and only real values of the independent variable are considered.

In point of method, we make use of two general theorems arising from Dini's[¶] researches in the theory of linear differential equations. The statement of these theorems, with an outline of their proof, is to be found in Sec. I.

The equation of second order forms the subject of Sec. II. Although the researches of Horn,[**] Kneser,[††] Bôcher[‡‡] and others leave less to be done

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[*] *Journal für Mathematik*, Vol. CXXXVIII (1910), pp. 159–191.

[†] Horn has published several papers on this case of distinct roots. Important contributions have also been made by Poincaré, Kneser, Birkhoff and others.

[‡] The term "regular singular point" has here the meaning which is assigned to it by Bôcher. Cf. *Transactions of the American Mathematical Society*, Vol. I (1901), pp. 40–52.

[§] Bôcher, *loc. cit.*

[||] *Proceedings of the American Academy of Arts and Sciences*, Vol. XXXVIII (1903), pp. 339-370.

[¶] *Annali di Matematica*, Ser. 3, Vol. II (1898), pp. 297–324; *ibid.*, Vol. III (1899), pp. 125–183. The possibility of applying Dini's methods to the problem in hand was suggested to me by Prof. W. B. Ford.

[**] *Acta Mathematica*, Vol. XXIII (1900), pp. 171–202.

[††] *Journal für Mathematik*, Vol. CXVI (1896), pp. 178–212; *ibid.*, Vol. CXVII (1896), pp. 72–103; *ibid.*, Vol. CXX (1899), pp. 267–275.

[‡‡] Bôcher, *loc. cit.*

upon this equation than upon those of the third and higher orders, it is believed that the discussion, from the standpoint of the Dini theory, is of sufficient interest to warrant its insertion in condensed form. We begin with a brief treatment of the case of distinct roots. While the results for this case are not new, it seems best to exemplify the method by applying it first to this simple problem, so that in the subsequent discussion of more complicated cases many details may be omitted. For the equation under consideration, the only case of repeated roots that offers any difficulty is that in which the point at infinity is a "regular" singular point, and this, as noted above, has been studied by Bôcher. However, on account of his more general hypotheses he does not obtain an asymptotic solution in Poincaré's * sense, but only the dominant term of such a solution, so that the present results are a step in advance.

Questions of more interest arise in connection with the equation of third order, which is discussed at length in Sec. III. This discussion, when compared with that for the equation of second order, is of distinctly greater moment, not only because the results possess greater novelty, but because the methods used are much more suggestive in pointing the way toward a solution of the general problem. This is especially true of the case in which the characteristic equation has a simple and a double root.

I.   TWO GENERAL THEOREMS ON LINEAR DIFFERENTIAL EQUATIONS.†

Suppose that, in the differential equation

$$y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_n(x) y = 0, \tag{1}$$

the coefficients $a_1(x), \ldots, a_n(x)$ and their first n derivatives are continuous for all positive values of x sufficiently large. Let us choose n auxiliary functions $z_1, z_2, \ldots, z_n$ of x, which, with their first n derivatives, are continuous for large positive x, and such that for the same values of x the determinant

$$Q(x) = \begin{vmatrix} z_1 & z_1' & \ldots & z_1^{(n-1)} \\ z_2 & z_2' & \ldots & z_2^{(n-1)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ z_n & z_n' & \ldots & z_n^{(n-1)} \end{vmatrix} \tag{2}$$

never vanishes. Let $A_r(x)$ denote the minor of $Q(x)$ with respect to the element $z_r^{(n-1)}$. Also define

$$Z_r(x) = z_r a_n - (z_r a_{n-1})' + \ldots + (-1)^{n-1} (z_r a_1)^{(n-1)} + (-1)^n z_r^{(n)}, \tag{3}$$
$$r = 1, \ldots, n,$$

* *Acta Mathematica*, Vol. VIII (1886), p. 296.   † Cf. Dini, *loc. cit.*

and form the determinant

$$q(x, x_1) = \begin{vmatrix} z_1(x) & z_1'(x) & \dots & z_1^{(n-2)}(x) & Z_1(x_1) \\ z_2(x) & z_2'(x) & \dots & z_2^{(n-2)}(x) & Z_2(x_1) \\ \hdotsfor{5} \\ z_n(x) & z_n'(x) & \dots & z_n^{(n-2)}(x) & Z_n(x_1) \end{vmatrix}. \tag{4}$$

Let C_r denote an arbitrary constant, and place

$$K(x, x_1) = \frac{(-1)^{n-1} q(x, x_1)}{Q(x)}, \tag{5}$$

$$\left.\begin{aligned}
g_r(x) &= u_{r,0}(x) = v_{r,0}(x) = \frac{(-1)^{n-1} C_r A_r(x)}{Q(x)}, \qquad r=1, \dots, n, \\
u_{r,\lambda}(x) &= \int_a^x \int_a^{x_1} \dots \int_a^{x_\lambda-2} \int_a^{x_\lambda-1} K(x, x_1) K(x_1, x_2) \dots \\
&\qquad K(x_{\lambda-2}, x_{\lambda-1}) K(x_{\lambda-1}, x_\lambda) g_r(x_\lambda)\, dx_\lambda dx_{\lambda-1} \dots dx_2 dx_1, \\
v_{r,\lambda}(x) &= \int_x^\infty \int_{x_1}^\infty \dots \int_{x_{\lambda-2}}^\infty \int_{x_{\lambda-1}}^\infty K(x, x_1) K(x_1, x_2) \dots \\
&\qquad K(x_{\lambda-2}, x_{\lambda-1}) K(x_{\lambda-1}, x_\lambda) g_r(x_\lambda)\, dx_\lambda dx_{\lambda-1} \dots dx_2 dx_1.
\end{aligned}\right\} \tag{6}$$

Then we have

THEOREM A: *If a constant a can be found such that for all values of $x > a$ the series*

$$y_r(x) = \sum_{\lambda=0}^\infty u_{r,\lambda}(x)$$

satisfies the following conditions:

(a) *the series converges;*

(b) *the series defines a function $y_r(x)$ such that the series for $y_r(x_1)$ when multiplied by $q(x, x_1)$ may be integrated term by term with respect to x_1 from a to x;*

then for such values of x the function $y_r(x)$ is an integral of (1).

THEOREM B: *If for values of x greater than some constant, the series*

$$y_r(x) = \sum_{\lambda=0}^\infty v_{r,\lambda}(x) \tag{7}$$

satisfies the following conditions:

(a) *the series converges;*

(b) *the series for $y_r(x_1)$ when multiplied by $q(x, x_1)$ may be integrated term by term with respect to x_1 from x to ∞ ;*

(c) *the series defines a function $y_r(x)$ such that each of the integrals*

$$\int_x^\infty y_r(x) Z_s(x)\, dx, \qquad s=1, \dots, n,$$

has a meaning;

then for such values of x the function $y_r(x)$ is an integral of (1).

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To prove Theorem B, place

$$
\left.
\begin{aligned}
&p_{s,0} = z_s, \\
&p_{s,1} = z_s a_1 - z'_s = z_s a_1 - p'_{s,0}, \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
&p_{s,n-1} = z_s a_{n-1} - p'_{s,n-2}, \qquad s = 1, 2, \ldots, n;
\end{aligned}
\right\}
\tag{8}
$$

$$
\Phi_s(x) = \int_x^\infty y_r Z_s \, dx, \qquad s = 1, 2, \ldots, r-1, r+1, \ldots, n,
$$
$$
\Phi_r(x) = \int_x^\infty y_r Z_r \, dx + C_r;
$$

$$
\Delta_r(x) =
\begin{vmatrix}
p_{1,0} & p_{1,1} & \cdots & p_{1,n-2} & \Phi_1(x) \\
p_{2,0} & p_{2,1} & \cdots & p_{2,n-2} & \Phi_2(x) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
p_{n,0} & p_{n,1} & \cdots & p_{n,n-2} & \Phi_n(x)
\end{vmatrix};
$$

$$
\Delta(x) =
\begin{vmatrix}
p_{1,0} & p_{1,1} & \cdots & p_{1,n-1} \\
p_{2,0} & p_{2,1} & \cdots & p_{2,n-1} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
p_{n,0} & p_{n,1} & \cdots & p_{n,n-1}
\end{vmatrix}
= (-1)^{\frac{n(n-1)}{2}} Q(x).
$$

Now by condition (c) the series (7) may be written

$$
y_r(x) = g_r(x) + \int_x^\infty y_r(x_1) K(x, x_1) \, dx_1.
\tag{9}
$$

By substituting the values of $g_r(x)$ and $K(x, x_1)$ in (9), we find

$$
y_r(x) = \frac{\Delta_r(x)}{\Delta(x)},
$$

so that it suffices for our proof to show that this function is an integral of (1).

To do this, consider the system of n functions $\eta_0, \eta_1, \ldots, \eta_{n-1}$, each defined for all values of x sufficiently large by means of the following system of n linear equations:

$$
p_{s,0}\eta_{n-1} + p_{s,1}\eta_{n-2} + \ldots + p_{s,n-1}\eta_0 = \Phi_s(x), \qquad s = 1, 2, \ldots, n.
\tag{10}
$$

We have at once

$$
\eta_0 = \frac{\Delta_r(x)}{\Delta(x)} = y_r.
\tag{11}
$$

Upon differentiating equations (10) with respect to x, and making use of (8) and (11), we find

$$
z_s \theta + p'_{s,0}\theta_1 + p'_{s,1}\theta_2 + \ldots + p'_{s,n-2}\theta_{n-1} = 0, \qquad \cdot s = 1, 2, \ldots, n,
\tag{12}
$$

where

$$
\theta = \eta'_{n-1} + a_1(x)\eta'_{n-2} + \ldots + a_{n-1}(x)\eta'_0 + a_n(x)\eta_0,
\tag{13}
$$

$$
\left.
\begin{aligned}
&\theta_1 = \eta_{n-1} - \eta'_{n-2}, \\
&\theta_2 = \eta_{n-2} - \eta'_{n-3}, \\
&\qquad \cdots\cdots\cdots\cdots, \\
&\theta_{n-1} = \eta_1 - \eta'_0.
\end{aligned}
\right\}
\tag{14}
$$

The system (12) consists of n homogeneous linear equations in the n quantities $\theta, \theta_1, \ldots, \theta_{n-1}$. By virtue of the relation

$$p'_{t,s} = z_t a_{s+1} - p_{t,s+1}, \qquad t = 1, 2, \ldots, n; \ s = 0, 1, \ldots, n-2,$$

the discriminant of the system reduces at once to $(-1)^{n-1}Q(x)$, and hence by our hypotheses does not vanish for any value of x under consideration, whence

$$\theta \equiv \theta_1 \equiv \ldots \equiv \theta_{n-1} \equiv 0,$$

or, by (11) and (14),

$$\eta_s = y_r^{(s)}, \qquad s = 1, 2, \ldots, n.$$

Substituting in (13), we find

$$y_r^{(n)} + a_1(x) y_r^{(n-1)} + \ldots + a_n(x) y_r \equiv 0,$$

which was to be proved.

The proof of Theorem A follows similar lines, and may be omitted.

II. THE EQUATION OF SECOND ORDER.

In the differential equation

$$y'' + a(x)y' + b(x)y = 0, \tag{15}$$

suppose that for sufficiently large positive real values of x we may write

$$a(x) \sim x^k \left[a_0 + \frac{a_1}{x} + \ldots \right],$$

$$b(x) \sim x^{2k} \left[b_0 + \frac{b_1}{x} + \ldots \right],$$

k being 0 or a positive integer, and suppose also that $a'(x)$ has an asymptotic development.

1. Distinct Roots.

Consider first the case in which the roots m_1, m_2 of the characteristic equation

$$m^2 + a_0 m + b_0 = 0$$

are distinct.

Let us choose two auxiliary functions $z_1(x)$, $z_2(x)$ (cf. Sec. I) of the form

$$z_r = e^{-f_r(x) + \phi_r(x)} x^{-a_{r,0}}, \qquad r = 1, 2 \tag{16}$$

where

$$f_r(x) = \frac{m_r x^{k+1}}{k+1} + \frac{\alpha_{r,-k} x^k}{k} + \frac{\alpha_{r,-k+1} x^{k-1}}{k-1} + \ldots + \alpha_{r,-1} x,$$

and

$$\phi_r(x) = \frac{\alpha_{r\,1}}{x} + \frac{\alpha_{r\,2}}{2x^2} + \ldots + \frac{\alpha_{r,s-1}}{(s-1)x^{s-1}}.$$

The undetermined coefficients $\alpha_{r,-k}, \ldots, \alpha_{r,s-1}$ and the arbitrary integer s will be selected presently.

Upon forming $Z_r(x)$ as given by (3), we find *

$$Z_r(x) = z_r x^{2k} \left[\frac{\zeta_{r,1}}{x} + \frac{\zeta_{r,2}}{x^2} + \ldots + \frac{\zeta_{r,s+k-1}}{x^{s+k-1}} + \frac{\zeta_{r,s+k}}{x^{s+k}} P_{1,r}(x) \right], \qquad r=1, 2,$$

in which $\zeta_{r,1}, \zeta_{r,2}, \ldots, \zeta_{r,s+k}$ are certain easily determined functions of $\alpha_{r,-k}, \ldots, \alpha_{r,s-1}$.

Now let us place

$$\zeta_{r,1} = \zeta_{r,2} = \ldots = \zeta_{r,s+k-1} = 0, \quad \zeta_{r,s+k} = (-1)^r(m_2 - m_1), \qquad r=1, 2,$$

and determine the coefficients † $\alpha_{r,-k}, \ldots, \alpha_{r,s-1}$ by means of this system of equations. Under our present hypotheses the equations can always be solved, and the functions z_1, z_2 thus determined will be unique. Let our notation be such that ‡

$$R[f_1(x)] \leq R[f_2(x)],$$

when x is sufficiently large.

Put

$$\alpha_{1,0} + \alpha_{2,0} = \alpha_0, \quad \alpha_{r,0} - k = \rho_r, \quad f_1(x) + f_2(x) = f(x).$$

Then we may write, by (4), (2), (6) and (5), respectively,

$$q(x, x_1) = \sum_{r=1}^{2} e^{f_r(x) - f(x)} x^{\alpha_{r,0} - \alpha_0} P_{2,r}(x) e^{-f_r(x_1)} x_1^{-\rho_r - s} P_{1,r}(x_1),$$

$$Q(x) = (m_1 - m_2) e^{-f(x)} x^{-\alpha_0 + k} P_1(x),$$

$$g_r(x) = \frac{C_r(-1)^r}{m_1 - m_2} e^{f_r(x)} x^{\rho_r} P_{3,r}(x), \qquad r=1, 2,$$

$$K(x, x_1) = \sum_{r=1}^{2} e^{f_r(x)} x^{\rho_r} P_{3,r}(x) e^{-f_r(x_1)} x_1^{-\rho_r - s} P_{1,r}(x_1).$$

An arbitrarily large integer p having been fixed, let us choose s so large that

$$s > p + 2 + 2|\alpha_{1,0}| + 2|\alpha_{2,0}|;$$

also take $C_r = (-1)^r(m_1 - m_2)$ and form the function $v_{1,\lambda}(x)$ occurring in Theorem B. Then the function y_1 of that theorem takes the form

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*Throughout the present work the symbol $P(x)$ (generally written with subscripts) denotes a function expressible in the form

$$P(x) = 1 + \frac{A_1}{x} + \ldots + \frac{A_p + \epsilon_p(x)}{x^p}, \quad \lim_{x = \infty} \epsilon_p(x) = 0,$$

p being an arbitrary integer.

†As soon as $\alpha_{1,0}, \alpha_{2,0}$ are determined, s is selected to fulfil a condition pointed out later.

‡Throughout the work, $R[x]$ denotes the real part of x.

$$y_1 = e^{f_1(x)} x^{\rho_1} P_{3,r}(x) + \sum_{\lambda=1}^{\infty} v_{1,\lambda}(x).$$

Upon recalling the way in which s was chosen, together with the fact that

$$R[f_1(x) - f_2(x)] \leqq 0,$$

it appears that the function $|K(x, x_1) g_1(x_1)|$ is a monotonically decreasing function of x_1, whence

$$\int_{x_{\lambda-1}}^{\infty} |K(x_{\lambda-1}, x_\lambda) g_1(x_\lambda)| dx_\lambda < |K(x_{\lambda-1}, x_{\lambda-1}) g_1(x_{\lambda-1})| < 4 |e^{f_1(x_{\lambda-1})} x_{\lambda-1}^{\rho_1-(p+2)}|.$$

Further,

$$\int_{x_{\lambda-2}}^{\infty} \int_{x_{\lambda-1}}^{\infty} |K(x_{\lambda-2}, x_{\lambda-1})| K(x_{\lambda-1}, x_\lambda) g_1(x_\lambda)| dx_\lambda dx_{\lambda-1} < 4^2 |e^{f_1(x_{\lambda-2})} x_{\lambda-2}^{\rho_1-2(p+2)}|.$$

Proceeding in this way, we find that

$$|v_{1,\lambda}(x)| < \frac{4^\lambda}{(p+1)^\lambda x^{\lambda(p+1)}} |e^{f_1(x)} x^{\rho_1}|,$$

and therefore that

$$\left| \sum_{\lambda=1}^{\infty} [v_{1,\lambda}(x)] \right| < \frac{8}{p+1} \cdot \frac{1}{x^{p+1}} |e^{f_1(x)} x^{\rho_1}|, \tag{17}$$

when x is sufficiently large. This inequality being established, it follows very readily that all the conditions of Theorem B are satisfied, and hence the function y_1 is an actual solution of (15).

As a consequence of (17), we may write

$$y_1 = e^{f_1(x)} x^{\rho_1} P_{4,1}(x), \tag{18}$$

and therefore

$$y_1 \sim e^{f_1(x)} x^{\rho_1} \left[1 + \frac{A_{1,1}}{x} + \dots \right]. \tag{19}$$

It is not difficult to show that y_1' and y_1'' also possess asymptotic developments, which may of course be found by differentiating (19).

Now if $R[f_2(x)] = R[f_1(x)]$, we may write at once by the same argument

$$y_2 = e^{f_2(x)} x^{\rho_2} P_{4,2}(x). \tag{20}$$

If, however, $R[f_1(x)] < R[f_2(x)]$, the argument fails, since some of the improper integrals in $v_{2,\lambda}(x)$ cease to have a meaning. To obtain a solution in this case, it will be convenient to revise our system of auxiliary functions, and then apply Theorem A.

In the adjoint equation

$$z'' - (az)' + bz = 0, \tag{21}$$

corresponding to (15), the coefficients satisfy all the hypotheses of this section. Thus (21) has a solution $\bar{z}_2$ of the form (18), which is found by direct computation to be

$$\bar{z}_2 = e^{-f_2(x)} x^{-a_{2,0}} P_{5,2}(x).$$

In the system (16) let us replace z_2 by this function $\bar{z}_2$, thus making $Z_2(x) \equiv 0$. With this change we find that

$$K(x, x_1) = e^{f_1(x)} x^{\rho_1} P_{6,1}(x) e^{-f_1(x_1)} x_1^{-\rho_1 - s} P_{1,1}(x_1),$$

while $g_2(x)$ is in essentials unchanged.

Now form the function $u_{2,\lambda}(x)$ of Theorem A. Then by that theorem we find

$$y_2 = e^{f_2(x)} x^{\rho_2} P_{4,2}(x) + \sum_{\lambda=1}^{\infty} u_{2,\lambda}(x).$$

Now upon recalling that

$$R[f_1(x) - f_2(x)] < 0,$$

we see that, if a be chosen sufficiently large,

$$\left| \sum_{\lambda=1}^{\infty} [u_{2,\lambda}(x)] \right| < \frac{4}{p+1} \cdot \frac{1}{x^{p+1}} \left| e^{f_2(x)} x^{\rho_2} \right|,$$

whence both the conditions of Theorem A are satisfied, and y_2 is a solution of (15). It may evidently be expressed in the form (20).

2.   *Equal Roots.*

To consider the case $m_1 = m_2$ it will be convenient to reduce the differential equation to the form

$$y'' + b(x)y = 0. \tag{22}$$

Now if $m_1 = m_2$, we must have either

$$b(x) \sim x^{2k} \left[\frac{b_1}{x} + \frac{b_2}{x^2} + \cdots \right], \qquad b_1 \neq 0, \tag{23}$$

or

$$b(x) \sim \frac{b_2}{x^2} + \frac{b_3}{x^3} + \cdots \cdots \tag{24}$$

When $b(x)$ has the form (23), we may reduce the problem to the case of distinct roots by merely introducing* $t = x^{\frac{1}{2}}$ as a new independent variable.

It remains only to study the case in which (24) holds. Here the point at infinity is a regular singular point† of the differential equation, so that the problem is closely related to that solved by Bôcher.† Hence only the briefest of discussions is necessary.

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*Cf. Kneser, *loc. cit.* (third paper), p. 275.   † Cf. Bôcher, *loc. cit.*

Let us choose two auxiliary functions

$$z_1 = x^{\rho_1}\left[1 + \frac{\alpha_1}{x} + \ldots + \frac{\alpha_{s-1}}{x^{s-1}}\right],$$

$$z_2 = x^{\rho_2}\left[1 + \frac{\beta_1}{x} + \ldots + \frac{\beta_{s-1}}{x^{s-1}}\right] + Bz_1 \log x,$$

and determine the constants by the use of $Z_1(x)$ and $Z_2(x)$, as in the case of distinct roots. We find $B \neq 0$ whenever $\rho_2 = \rho_1$, and, in general, also when $\rho_2 - \rho_1$ is a positive integer, but otherwise $B = 0$. By use of Theorem B we may obtain the asymptotic forms of two independent solutions y_1, y_2 of (22). The developments are of course *formally* identical with those obtained when the point at infinity is a regular point in the sense of the Fuchs theory.

The results for the equation of second order may be summarized in

THEOREM I: *In the differential equation*

$$y'' + b(x)y = 0, \tag{22}$$

suppose that $b(x)$ *is a real or complex function developable asymptotically, for large real positive values of* x, *in the form*

$$b(x) \sim x^{2k}\left[b_0 + \frac{b_1}{x} + \ldots\right],$$

where k *is* 0 *or a positive integer. Then, for the same values of* x, *equation* (22) *possesses two linearly independent solutions* y_1, y_2 *such that*

(a) *if* $b_0 \neq 0$, *i. e., if the roots* m_1, m_2 *of the characteristic equation* $m^2 + b_0 = 0$ *are distinct, we may write*

$$y_r \sim e^{f_r(x)}x^{\rho_r}\left[1 + \frac{A_{r,1}}{x} + \ldots\right], \qquad r = 1, 2,$$

where

$$f_r(x) = \frac{m_r x^{k+1}}{k+1} + \frac{\alpha_{r,-k}x^k}{k} + \ldots + \alpha_{r,-1}x;$$

(b) *if* $b_0 = 0$, $b_1 \neq 0$, *we may write*

$$y_r \sim e^{f_r(x)}x^{\rho_r}\left[1 + \frac{A_{r,1}}{x} + \ldots + \frac{1}{x^{\frac{1}{2}}}\left(B_{r,0} + \frac{B_{r,1}}{x} + \ldots\right)\right], \qquad r = 1, 2,$$

where

$$f_r(x) = \frac{\alpha_{r,-2k-1}x^{k+\frac{1}{2}}}{k+\frac{1}{2}} + \frac{\alpha_{r,-2k}x^k}{k} + \ldots + \frac{\alpha_{r,-1}x^{\frac{1}{2}}}{\frac{1}{2}};$$

(c) *if* $k = b_0 = b_1 = 0$, *we may write, in general,*

$$y_r \sim x^{\rho_r}\left[1 + \frac{A_{r,1}}{x} + \ldots\right], \qquad r = 1, 2;$$

(d) *but if $\rho_2 = \rho_1$ or, in general, if $\rho_2 - \rho_1$ is a positive integer, we have*

$$y_1 \sim x^{\rho_1}\left[1 + \frac{A_{1,1}}{x} + \ldots\right],$$

$$y_2 \sim y_1 \log x + x^{\rho_2}\left[A_{2,0} + \frac{A_{2,1}}{x} + \ldots\right].$$

III. THE EQUATION OF THIRD ORDER.

Take for consideration a differential equation of the third order, which we shall suppose reduced to the form

$$y''' + b(x)y' + c(x)y = 0. \tag{25}$$

Let $b(x)$ and $c(x)$ be developable, when x is large, in the form

$$b(x) \sim x^{2k}\left[b_0 + \frac{b_1}{x} + \ldots\right],$$

$$c(x) \sim x^{3k}\left[c_0 + \frac{c_1}{x} + \ldots\right],$$

and suppose that $b'(x)$ also has an asymptotic development.

If the roots m_1, m_2, m_3 of the characteristic equation

$$m^3 + b_0 m + c_0 = 0$$

are distinct, the asymptotic solutions are well-known.* We therefore pass at once to the cases of multiple roots.

1. *A Simple and a Double Root.*

Suppose first that two of the roots are equal—say $m_1 \neq m_2 = m_3$.

Let us select three auxiliary functions

$$z = e^{-f_r(x) + \phi_r(x)} x^{-a_{r,0}}, \qquad r = 1, 2, 3,$$

in which

$$f_r(x) = \frac{m_r x^{k+1}}{k+1} + \frac{\alpha_{r,-2k-1} x^{k+\frac{1}{2}}}{k+\frac{1}{2}} + \frac{\alpha_{r,-2k} x^k}{k} + \ldots + \frac{\alpha_{r,-1} x^{\frac{1}{2}}}{\frac{1}{2}},$$

$$\phi_r(x) = \frac{\alpha_{r,1}}{\frac{1}{2} x^{\frac{1}{2}}} + \frac{\alpha_{r,2}}{x} + \ldots + \frac{\alpha_{r,2s-2}}{(s-1) x^{s-1}}.$$

Upon forming $Z_r(x)$, we find that

$$Z_r(x) = z_r x^{3k}\left[\frac{\zeta_{r,1}}{x^{\frac{1}{2}}} + \frac{\zeta_{r,2}}{x} + \ldots + \frac{\zeta_{r,2s+2k-1}}{x^{s+k-\frac{1}{2}}} + \frac{\zeta_{r,2s+2k}}{x^{s+k}} P_{1,r}(x^{\frac{1}{2}})\right], \qquad r = 1, 2, 3,$$

$\zeta_{r,1}, \ldots, \zeta_{r,2s+2k}$ being certain easily formed functions of the undetermined constants in $f_r(x)$ and $\phi_r(x)$.

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* Cf. Horn, *Journal für Mathematik*, Vol. CXXXVIII (1910), pp. 159–191. The results are easily obtainable by the present theory.

Let us try to determine these constants by placing

$$\zeta_{r,1}=\ldots.=\zeta_{r,2s+2k-1}=0, \quad \zeta_{r,2s+2k}=\theta_r, \tag{27}$$

the constants θ_r being for the present unspecified. Inspection of the equations thus formed shows that the function z_1 corresponding to the simple root m_1 is always determinate, and that the same is true of one of the other auxiliary functions, say z_3. In determining z_2 by (27), difficulty may arise. Suppose that the first h coefficients of z_2, as given by (27), coincide with the corresponding ones in z_3, h being 0 or some positive integer. If $h \leq 2k$, z_2 may be determined by (27). If $h = 2k+1$, there is no difficulty unless the difference $\alpha_{3,0} - \alpha_{2,0}$ is an integer, in which case the equations in general become illusory. Finally, if $h > 2k+1$, i. e., $\alpha_{3,0} = \alpha_{2,0}$, z_2 coincides with z_3 throughout.

We consider first the ordinary case in which equations (27) serve to determine three definite, distinct functions of the form (26).

Place $f_1(x) + f_2(x) + f_3(x) = f(x)$,

$$\delta_1 = \alpha_{3,-2k-1+h} - \alpha_{2,-2k-1+h},$$
$$\delta_2 = -\delta_3 = m_1 - m_2,$$
$$\delta = \delta_1 \delta_2 \delta_3,$$
$$\theta_r = \frac{\delta}{\delta_r}, \qquad r = 1, 2, 3,$$
$$\beta_1 = 0, \quad \beta_2 = \beta_3 = \frac{h+1}{2},$$
$$\rho_r = \alpha_{r,0} - 2k + \beta_r.$$

Then we get

$$Q(x) = \delta e^{-f(x)} x^{-a_0 + 3k - \frac{h+1}{2}} P_1(x^{\frac{1}{2}}),$$

$$g_r(x) = C_r \frac{\delta_r}{\delta} e^{f_r(x)} x^{\rho_r} P_{2,r}(x^{\frac{1}{2}}), \qquad r = 1, 2, 3,$$

$$K(x, x_1) = \sum_{r=1}^{3} e^{f_r(x)} x^{\rho_r} P_{2,r}(x^{\frac{1}{2}}) e^{-f_r(x_1)} x_1^{\beta_r - \rho_r - s} P_{1,r}(x_1^{\frac{1}{2}}). \tag{28}$$

Consider first the case in which the functions $f_r(x)$ have distinct real parts. Suppose for definiteness that

$$R[f_1(x)] < R[f_2(x)] < R[f_3(x)]. \tag{29}$$

Take $C_r = \theta_r$, and form $v_{1,\lambda}(x)$. Then by Theorem B we have

$$y_1 = e^{f_1(x)} x^{\rho_1} P_{2,r}(x^{\frac{1}{2}}) + \sum_{\lambda=1}^{\infty} v_{1,\lambda}(x). \tag{30}$$

In $v_{1,\lambda}(x)$, we have

$$x_\lambda \geq x_{\lambda-1} \geq \ldots. \geq x_1 \geq x.$$

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Further, by (29),

$$R[f_1(x) - f_r(x)] < 0, \qquad r = 2, 3.$$

Whence, if we choose

$$s > p + 2 + h + 1 + \sum_{r=1}^{3} |R[\rho_r]|,$$

p being, as usual, an arbitrary integer, it appears by argument like that used in Sec. II that

$$|v_{1,\lambda}(x)| < \frac{2 \cdot 3^\lambda}{(p+1)^\lambda x^{\lambda(p+1)}} |e^{f_1(x)} \dot{x}^{\rho_1}|.$$

Thus

$$\left| \sum_{\lambda=1}^{\infty} [v_{1,\lambda}(x)] \right| < \frac{4 \cdot 3}{p+1} \cdot \frac{1}{x^{p+1}} |e^{f_1(x)} x^{\rho_1}|.$$

This inequality once established, it is easy to show that y_1 satisfies the conditions of Theorem B, and is thus a solution of (25). Further, we may evidently write

$$y_1 = e^{f_1(x)} x^{\rho_1} P_{3,1}(x^{\frac{1}{2}}). \tag{31}$$

To obtain a second solution, let us employ a device similar to that used in Sec. II. The adjoint equation

$$z''' + (bz)' - cz = 0 \tag{32}$$

may be shown by (31) to have a solution $\bar{z}_3$ expressible in the form

$$\bar{z}_3 = e^{-f_3(x)} x^{-a_{3,0}} P_{4,3}(x^{\frac{1}{2}}).$$

We shall replace the function z_3 by this $\bar{z}_3$, thus making $Z_3(x) \equiv 0$, so that in $K(x, x_1)$ r takes only the values 1, 2.

Form $u_{3,\lambda}(x)$: Then by Theorem A

$$y_3 = e^{f_3(x)} x^{\rho_3} P_{5,3}(x^{\frac{1}{2}}) + \sum_{\lambda=1}^{\infty} u_{3,\lambda}(x).$$

Now we have in the present instance

$$x_\lambda \leqq x_{\lambda-1} \leqq \ldots \leqq x_1 \leqq x,$$

and

$$R[f_3(x) - f_r(x)] > 0, \qquad r = 1, 2,$$

so that

$$|u_{3,\lambda}(x)| < \frac{2 \cdot 2^\lambda}{(p+1)^\lambda x^{\lambda(p+1)}} |e^{f_3(x)} x^{\rho_3}|,$$

and

$$\left| \sum_{\lambda=1}^{\infty} [u_{3,\lambda}(x)] \right| < \frac{2 \cdot 4}{p+1} \cdot \frac{1}{x^{p+1}} |e^{f_3(x)} x^{\rho_3}|,$$

whence y_3 is a solution of (25). It may be put in the form

$$y_3 = e^{f_3(x)} x^{\rho_3} P_{3,3}(x^{\frac{1}{2}}).$$

We see now that (32) has a solution

$$\bar{z}_1 = e^{-f_1(x)} x^{-a_{1,0}} P_{4,1}(x^{\frac{1}{2}}).$$

Take this as an auxiliary function instead of z_1, so that $Z_1(x) \equiv 0$, and $K(x, x_1)$ reduces to the form

$$K(x, x_1) = e^{f_2(x)} x^{\rho_2} P_{6,2}(x^{\frac{1}{2}}) e^{-f_2(x_1)} x_1^{\beta_2 - \rho_2 - s} P_{7,2}(x_1)^{\frac{1}{2}}.$$

Upon writing out y_2 by Theorem B, we find by argument now familiar that

$$y_2 = e^{f_2(x)} x^{\rho_2} P_{3,2}(x^{\frac{1}{2}}).$$

This evidently disposes of the case in which the functions $f_r(x)$ have distinct real parts. Only very slight changes in the argument are needed if two real parts are equal, while if all three are equal the three solutions may be written down by Theorem B at once.

By direct substitution in (25), it appears that all the terms in y_1 involving fractional powers of x disappear.

We return now to the exceptional cases noted above, in which equations (27) do not serve to determine three functions of the form (26).

Suppose equations (27) become illusory, in which case $\alpha_{3,0} - \alpha_{2,0}$ must be an integer. Take

$$\left. \begin{aligned} z_1 &= e^{-f_1(x) + \phi_1(x)} x^{-a_{1,0}}, \\ z_2 &= e^{-f_2(x) + \phi_2(x)} x^{-a_{2,0}} + \beta z_3 \log x, \qquad \beta \neq 0, \\ z_3 &= e^{-f_2(x) + \phi_3(x)} x^{-a_{3,0}}, \end{aligned} \right\} \qquad (33)$$

in which

$$f_r(x) = \frac{m_r x^{k+1}}{k+1} + \frac{\alpha_{r,-k} x^k}{k} + \dots + \alpha_{r,-1} x, \qquad r = 1, 2,$$

$$\phi_r(x) = \frac{\alpha_{r,1}}{x} + \frac{\alpha_{r,2}}{2x^2} + \dots + \frac{\alpha_{r,s-1}}{(s-1)x^{s-1}}, \qquad r = 1, 2, 3.$$

The constants can now be determined without trouble in the usual way. Placing

$$\begin{aligned} \delta_1 &= \alpha_{3,0} - \alpha_{2,0}, \\ \delta_2 &= -\delta_3 = m_1 - m_2, \\ \delta &= \delta_1 \delta_2 \delta_3, \\ \rho_1 &= \alpha_{1,0} - 2k, \quad \rho_r = \alpha_{r,0} - k + 1, \qquad r = 2, 3, \end{aligned}$$

we find

$$\left. \begin{aligned} g_r(x) &= C_r \frac{\delta_r}{\delta} e^{f_r(x)} x^{\rho_r} P_{8,r}(x), \qquad r = 1, 2, \\ g_3(x) &= C_3 \frac{\delta_3}{\delta} e^{f_2(x)} x^{\rho_3} P_{8,3}(x) + \beta g_2(x) \log x, \end{aligned} \right\} \qquad (34)$$

$$Z_1(x) = e^{-f_1(x)}x^{-\alpha_{1,0}+2k-s}P_{9,1}(x),$$
$$Z_2(x) = e^{-f_2(x)}x^{-\alpha_{2,0}+2k-s}P_{9,2}(x) + \beta z_3 \log x,$$
$$Z_3(x) = e^{-f_2(x)}x^{-\alpha_{3,0}+2k-s}P_{9,3}(x),$$
$$K(x, x_1) = \sum_{r=1}^{3} g_r(x)Z_r(x_1).$$

By familiar reasoning it may be shown that (25) has three solutions of the form

$$\left.\begin{aligned} y_r &= e^{f_r(x)}x^{\rho_r}P_{10,r}(x), \qquad r=1, 2, \\ y_3 &= y_2 \log x + Be^{f_2(x)}x^{\rho_3}P_{10,3}(x), \end{aligned}\right\} \qquad (35)$$

where B is a constant.

There remains only the case in which the functions z_2 and z_3, as given by (27), are identical.

Take the auxiliary functions as in (33). We find that now $\alpha_{3,0}=\alpha_{2,0}$, $\alpha_{3,1} \neq \alpha_{2,1}$. If we place

$$\delta_1 = \alpha_{3,1} - \alpha_{2,1},$$
$$\delta_2 = -\delta_3 = m_1 - m_2,$$
$$\delta = \delta_1\delta_2\delta_3,$$

the functions $g_r(x)$ have the same form as in (34), and the three integrals y_r take the form (35), except that now $\rho_3 = \rho_2$.

2.   *A Triple Root.*

If $m_1 = m_2 = m_3$, we have either

$$\text{(a)} \quad \begin{cases} b(x) \sim x^{2k}\left[\dfrac{b_1}{x} + \dfrac{b_2}{x^2} + \dots\right], \\ c(x) \sim x^{3k}\left[\dfrac{c_1}{x} + \dfrac{c_2}{x_2} + \dots\right], \end{cases}$$

where b_1, c_1, c_2 are not all 0; or else

$$\text{(b)} \quad \begin{cases} b(x) \sim \dfrac{b_2}{x^2} + \dfrac{b_3}{x^3} + \dots, \\ c(x) \sim \dfrac{c_3}{x^3} + \dfrac{c_4}{x^4} + \dots. \end{cases}$$

In (a), if $c_1 \neq 0$ or if $b_1 = c_1 = 0$, $c_2 \neq 0$, we need only introduce the new independent variable $t = x^{\frac{1}{3}}$ in order to reduce the problem to the case of distinct roots.

If $c_1 = 0$, $b_1 \neq 0$, a similar reduction results if the new variable $t = x^{\frac{1}{2}}$ be introduced. One of the solutions is found to proceed in powers of x, the other two in powers of $x^{\frac{1}{2}}$.

There remains only (b). This case may be disposed of by Theorem B. On account of the closely related discussion by Dunkel, mentioned above, together with the formal analogy between this problem and that in which the point at infinity is a regular point in the ordinary sense, we content ourselves with a mere statement of results, in the theorem below.

In summary, our results for the equation of the third order are as follows:[†]

THEOREM II: *In the differential equation*

$$y''' + b(x)y' + c(x)y = 0, \tag{25}$$

suppose that $b(x)$ and $c(x)$ are real or complex functions developable asymptotically, when x is large, real and positive, in the form

$$b(x) \sim x^{2k}\left[b_0 + \frac{b_1}{x} + \dots\right],$$

$$c(x) \sim x^{3k}\left[c_0 + \frac{c_1}{x} + \dots\right],$$

where k is 0 or a positive integer, and suppose that $b'(x)$ also has an asymptotic development. Then for the same values of x equation (25) has three linearly independent solutions y_1, y_2, y_3 possessing asymptotic developments as follows:

(a) *If the roots m_1, m_2, m_3 of the characteristic equation*

$$m^3 + b_0 m + c_0 = 0$$

are distinct, we may write

$$y_r \sim e^{f_r(x)} x^{\rho_r}\left[1 + \frac{A_{r,1}}{x} + \dots\right], \qquad r = 1, 2, 3,$$

where

$$f_r(x) = \frac{m_r x^{k+1}}{k+1} + \frac{\alpha_{r,-k}x^k}{k} + \dots + \alpha_{r,-1}x.$$

(b) *If $m_1 \neq m_2 = m_3$, we may write in general*

$$y_1 \sim e^{f_1(x)} x^{\rho_1}\left[1 + \frac{A_{1,1}}{x} + \dots\right],$$

$$y_r \sim e^{f_r(x)} x^{\rho_r}\left[1 + \frac{A_{r,1}}{x} + \dots + \frac{1}{x^{\frac{1}{2}}}\left(B_{r,0} + \frac{B_{r,1}}{x} + \dots\right)\right], \qquad r = 2, 3,$$

where $f_1(x)$ has the same form as in (a) *and*

$$f_r(x) = \frac{m_r x^{k+1}}{k+1} + \frac{\alpha_{r,-2k-1}x^{k+\frac{1}{2}}}{k+\frac{1}{2}} + \frac{\alpha_{r,-2k}x^k}{k} + \dots + \frac{\alpha_{r,-1}x^{\frac{1}{2}}}{\frac{1}{2}}, \quad r = 2, 3;$$

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*Cf. Dunkel, *loc. cit.*

[†] The results for the case of distinct roots are included merely for completeness.

(c) *but if* $\rho_3 = \rho_2$, *or in general if* $\rho_3 - \rho_2$ *is a positive integer, we have*

$$y_r \sim e^{f_r(x)} x^{\rho_r} \left[1 + \frac{A_{r,1}}{x} + \ldots \right], \qquad r = 1, 2,$$

$$y_3 \sim y_2 \log x + e^{f_2(x)} x^{\rho_3} \left[A_{3,0} + \frac{A_{3,1}}{x} + \ldots \right],$$

where $f_1(x)$ *and* $f_2(x)$ *have the same form as in* (a).

(d) *If* $m_1 = m_2 = m_3$ *and either* $c_1 \neq 0$ *or* $b_1 = c_1 = 0$, $c_2 \neq 0$, *we may write*

$$y_r \sim e^{f_r(x)} x^{\rho_r} \left[1 + \frac{A_{r,1}}{x} + \ldots + \frac{1}{x^{\frac{1}{3}}} \left(B_{r,0} \frac{B_{r,1}}{x} + \ldots \right) \right.$$
$$\left. + \frac{1}{x^{\frac{2}{3}}} \left(C_{r,0} + \frac{C_{r,1}}{x} + \ldots \right) \right], \quad r = 1, 2, 3,$$

where

$$f_r(x) = \frac{m_r x^{k+1}}{k+1} + \frac{\alpha_{r,-3k-2} x^{k+\frac{2}{3}}}{k+\frac{2}{3}} + \frac{\alpha_{r,-3k-1} x^{k+\frac{1}{3}}}{k+\frac{1}{3}} + \ldots + \frac{\alpha_{r,-1} x^{\frac{1}{3}}}{\frac{1}{3}} \ ;$$

(e) *if* $c_1 = 0$, $b_1 \neq 0$, y_1, y_2, y_3 *have expansions of the same form as in* (b);

(f) *if* $k = b_1 = c_1 = c_2 = 0$, *we may write*

$$y_1 \sim x^{\rho_1} \left[1 + \frac{A_{1,1}}{x} + \ldots \right],$$

$$y_2 \sim A y_1 \log x + x^{\rho_2} \left[A_{2,0} + \frac{A_{2,1}}{x} + \ldots \right],$$

$$y_2 \sim B y_1 \log^2 x + x^{\rho_2} \log x \left[B_{3,0} + \frac{B_{3,1}}{x} + \ldots \right]$$
$$+ x^{\rho_3} \left[A_{3,0} + \frac{A_{3,1}}{x} + \ldots \right].$$

This evidently covers all cases that may arise in connection with the equation of the third order.